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AN OBSERVATION CONCERNING SIGNAL TO NOISE RATIO PROPERTIES OF C--ETC(U)
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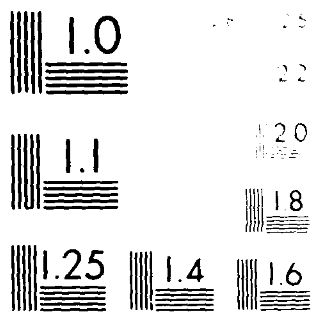
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AN OBSERVATION CONCERNING SIGNAL TO NOISE RATIO PROPERTIES OF CONTINUOUS AND DISCRETE TIME DETECTORS

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ABSTRACT

We present an approach toward improving the output signal to noise power ratio of a discrete time matched filter by employing a continuous time filter. The approach exploits the bandlimited nature of the unknown continuous time signal, and results in a filter which is robust to this inexact knowledge. We furthermore show that the robustness property does not severely compromise performance; in fact, the resulting filter possesses an output signal to noise power ratio which upper bounds that of the corresponding discrete time filter.

I. INTRODUCTION

The matched filter has been of practical interest for some time. This filter, which maximizes the output signal to noise power ratio, requires knowledge of the signal input for its design. In many cases it is reasonable to expect that the signal will be known at various discrete instants, thus admitting the design of the discrete time filter. Unfortunately, while it is reasonable to expect that the input signal will be known at discrete instants, it is another matter to assume that it will be known exactly as a closed form analytical expression over, for example, an interval of time, which would be necessary for the design of a matched filter in continuous time. Design of the continuous time filter is thus inhibited by such inexact knowledge of the signal input.

While the signal may be incompletely known, it is reasonable to expect that in many cases it could be modeled as bandlimited. If we furthermore

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assume the signal is known at a fixed number of instants, we might hope that a continuous time filter could be designed which is insensitive to the remaining inexactness in our knowledge of the signal. We also might hope that the resultant filter's performance would be good when compared to that of the discrete time filter.

In this paper we show that an approach realizing these aspirations is possible. We show that the resulting filter is extremely robust to inexact knowledge of the continuous time signal, and simultaneously possesses an output signal to noise power ratio which upper bounds that of the discrete time filter. These results show that in many cases it is desirable to conduct the filtering operation in continuous time, even though the signal may be known imperfectly.

II. DEVELOPMENT

Suppose the continuous time signal $s(t)$ is of finite energy and possesses a Fourier transform which vanishes outside the interval $[-W, W]$, where $W > 0$ is known, and suppose also that $s(t_i) = s_i$ for $i = 1, 2, \ldots, n$. We will employ the filter whose output at time T is $\int y(t)h(T-t)dt$, where $y(\cdot)$ is the received waveform and $h(\cdot)$ is the Green's function of the linear filter (all integrals, unless indicated otherwise, are taken over the entire real line). For the purposes of this discussion we will take T to be fixed and satisfy $T > t_i$ for all $i = 1, 2, \ldots, n$. The output signal to noise power ratio then becomes

$$\mathcal{R}(s, h) = \frac{[\int s(t)h(T-t)dt]^2}{\iint R(t-u)h(T-t)h(T-u)dt\, du},$$

where $R(\cdot)$ is the autocorrelation of the wide sense stationary noise. Note that the finite energy condition on $s(\cdot)$ allows us to place conditions on $h(\cdot)$ independent of $s(\cdot)$ so that $\mathcal{R}(\cdot, \cdot)$ is well defined. We will consider the class $\mathcal{H}$ of all finite energy $h(\cdot)$ such that $0 < \iint R(t-u)h(T-t)h(T-u)\, dt\, du < \infty$. Let $\mathcal{S}$ denote the class of admissable signals, i.e. finite energy signals possessing a Fourier transform which vanishes outside $[-W, W]$ and equaling s_i at t_i, $i = 1, 2, \ldots, n$. We then may interpret the filter design problem as essentially a choice of an element $h(\cdot)$ of $\mathcal{H}$ for a given element $s(\cdot)$ of $\mathcal{S}$.

In practice, of course, we do not know $s(\cdot)$; we only know that it belongs to $\mathcal{S}$. We must therefore accept a certain degree of inexactness of knowledge of $s(\cdot)$ in our decision procedure when choosing $h(\cdot)$. It would

obviously be beneficial to make the choice in a way which is insensitive to such inexact knowledge and which leads to good performance. In particular, we might hope that we could choose a filter with performance upper bounding that of the discrete time filter for any $s(\cdot) \in \mathcal{S}$.

The problems imposed by the presence of various types of inexact knowledge have been encountered in many other contexts. Frequently, it has been beneficial to employ approaches which are "robust" to the inexact knowledge. Such methods very often involve saddlepoint criteria, e.g. see [1 - 4]. The general success of this work thus motivates a saddlepoint approach to our somewhat different problem; we therefore seek $s(\cdot)$ and $h(\cdot)$ corresponding to $\inf_{s\in\mathcal{S}} \sup_{h\in\mathcal{H}} \mathcal{R}(s, h)$. Our first result shows how the filter may be designed by this criterion.

<u>Theorem 1:</u> Suppose $\det[(\mathrm{sinc}(W(\cdot-t_i))*R(\cdot))(t_j)]_{i,j} \neq 0$, and suppose $R(\cdot)$ has a Fourier transform that is positive on $[-W, W]$. Then we realize $\inf_{s\in\mathcal{S}} \sup_{h\in\mathcal{H}} \mathcal{R}(s, h) = \min_{s\in\mathcal{S}} \max_{h\in\mathcal{H}} \mathcal{R}(s, h)$ via $h(t) = \sum_{i=1}^{n} c_i \,\mathrm{sinc}(W(T-t_i-t))$ and $s(t) = [\sum_{i=1}^{n} c_i \,\mathrm{sinc}(W(\cdot-t_i))*R(\cdot)](t)$, where the constants c_i are chosen so that $s(t_i) = s_i$ for $i = 1, 2, \ldots, n$.

<u>Proof:</u> For any $h\in\mathcal{H}$, let $H(t) \stackrel{\Delta}{=} h(T-t)$, and denote the Fourier transforms of s, H, R by $\tilde{s}$, $\tilde{H}$, $\tilde{R}$ respectively. Then an argument similar to that of [5] may be used to show that for a fixed s, $\sup_{h\in\mathcal{H}} \mathcal{R}(s, h)$ is realized if and only if $\tilde{s}(\omega) = \tilde{R}(\omega)\tilde{H}(\omega)$ for almost all ω. We thus have $\tilde{H}(\omega) = \tilde{s}(\omega)\tilde{R}^{-1}(\omega)$ for almost all $\omega \in [-W, W]$. It is routine to verify that $\int s(t)H(t)dt$ is independent of the values $\tilde{H}(\cdot)$ takes on $[-W, W]^c$; moreover it is straightforward to show that $\iint R(t-u)H(t)H(u)dt\,du = \frac{1}{2\pi}\int \tilde{H}(\omega)\tilde{H}$[illegible]$)d\omega$, which is minimized for $\tilde{H}(\cdot)$ as above by choosing $\tilde{H}(\omega) = 0$ on $[-\infty$ [illegible]. Thus $\sup_{h\in\mathcal{H}} \mathcal{R}(s, h)$ is realized by choosing $\tilde{H}(\omega) = \begin{cases} \tilde{s}(\omega)\,\tilde{R}(\omega)^{-1} & \text{if } \omega \in [-W, W] \\ 0 & \text{otherwise} \end{cases}$. Choosing $h(\cdot)$ corresponding to the above $\tilde{H}(\cdot)$ for a given $s(\cdot)$, we thus must attempt to choose $s(\cdot)$ to realize $\inf_{s\in\mathcal{S}} \sup_{h\in\mathcal{H}} \mathcal{R}(s, h)$. Noting that scaling s allows us to hold the denominator of $\mathcal{R}(s, h)$ constant, we therefore minimize

$$\int s(t)H(t)dt-\lambda\iint H(t)H(u)R(t-u)dt\,du - \sum_{i=1}^{n} \lambda_i\, s(t_i) = -\frac{1}{(2\pi)^2}\int\int_{-W}^{W}\int_{-W}^{W} \tilde{s}(\omega)\tilde{s}(\omega)$$

$\tilde{R}^{-1}(\tilde{\omega})e^{j(\omega+\tilde{\omega})t}d\omega\, d\tilde{\omega}\, dt - \frac{\lambda}{(2\pi)^2}\int\int\int_{-W}^{W}\int_{-W}^{W} \tilde{s}(\omega)\tilde{s}(\tilde{\omega})\tilde{R}^{-1}(\omega)\tilde{R}^{-1}(\tilde{\omega})e^{j(\omega t+\tilde{\omega}u)}R(t-u)$

$d\omega\, d\tilde{\omega}\, dt\, du - \frac{1}{2\pi}\sum_{i=1}^{n}\lambda_i\int\tilde{s}(\omega)e^{j\omega t_i}\, d\omega$, where λ and $\{\lambda_i\}_{i=1}^{n}$ are undetermined Lagrange multipliers. Using a variational approach applied to $\tilde{s}(\cdot)$, we

minimize $J_{\tilde{s}}(\varepsilon) \overset{\Delta}{=} \frac{1}{(2\pi)^2}\int\int_{-W}^{W}\int_{-W}^{W}(\tilde{s}+\varepsilon\cdot\delta\tilde{s})(\omega)\cdot(\tilde{s}+\varepsilon\cdot\delta\tilde{s})(\tilde{\omega})\tilde{R}^{-1}(\tilde{\omega})e^{j(\omega+\tilde{\omega})t}d\omega\, d\tilde{\omega}\, dt$

$- \frac{\lambda}{(2\pi)^2}\int\int\int_{-W}^{W}\int_{-W}^{W}(\tilde{s}+\varepsilon\cdot\delta\tilde{s})(\omega)\cdot(\tilde{s}+\varepsilon\cdot\delta\tilde{s})(\tilde{\omega})\tilde{R}^{-1}(\omega)\tilde{R}^{-1}(\tilde{\omega})e^{j(\omega t+\tilde{\omega}u)}R(t-u)$

$d\omega\, d\tilde{\omega}\, dt\, du - \frac{1}{2\pi}\sum_{i=1}^{n}\lambda_i\int(\tilde{s}+\varepsilon\cdot\delta\tilde{s})(\omega)e^{j\omega t_i}d\omega$. Letting $J'_{\tilde{s}}(0) = 0$ for all $\delta\tilde{s}$,

we obtain $\frac{1}{(2\pi)^2}\int\int_{-W}^{W}\tilde{s}(\tilde{\omega})\tilde{R}^{-1}(\tilde{\omega})e^{j(\omega+\tilde{\omega})t}d\tilde{\omega}\, dt + \frac{1}{(2\pi)^2}\int\int_{-W}^{W}\tilde{s}(\omega)\tilde{R}^{-1}(\omega)e^{j(\omega+\tilde{\omega})t}$

$d\tilde{\omega}\, dt - \frac{\lambda}{(2\pi)^2}\int\int\int_{-W}^{W}\tilde{s}(\tilde{\omega})\tilde{R}^{-1}(\tilde{\omega})\tilde{R}^{-1}(\omega)e^{j(\omega t+\tilde{\omega}u)}R(t-u)d\tilde{\omega}\, dt\, du - \frac{\lambda}{(2\pi)^2}\int\int\int_{-W}^{W}\tilde{s}(\tilde{\omega})$

$\cdot\tilde{R}^{-1}(\tilde{\omega})\tilde{R}^{-1}(\omega)e^{j(\tilde{\omega}t+\omega u)}R(t-u)d\tilde{\omega}\, dt\, du - \frac{1}{2\pi}\sum_{i=1}^{n}\lambda_i e^{j\omega t_i} = 0$ for almost all

$\omega\in[-W, W]$. This implies via a straightforward calculation $\frac{1}{\pi}\tilde{s}(-\omega)\tilde{R}^{-1}(\omega) - \frac{\lambda}{2\pi}\tilde{s}(-\omega)\tilde{R}^{-1}(\omega) - \frac{1}{2\pi}\sum_{i=1}^{n}\lambda_i\, e^{j\omega t_i} = 0$ for almost all $\omega\in[-W, W]$. Thus $\tilde{s}(\omega) = \sum_{i=1}^{n}\tilde{c}_i\,\tilde{R}(\omega)\, e^{-j\omega t_i}\cdot I_{[-W,\, W]}(\omega)$ almost everywhere for some constants $\{\tilde{c}_i\}_{i=1}^{n}$, where $I_{[-W,\, W]}(\cdot)$ is the indicator function for the interval $[-W, W]$. We therefore obtain $s(t) = [\sum_{i=1}^{n} c_i\ \mathrm{sinc}(W(\cdot-t_i))*R(\cdot)](t)$ for some constants $\{c_i\}_{i=1}^{n}$.

In order to establish that this condition is sufficient, we note that a straightforward calculation shows $J_{\tilde{s}}(\varepsilon) = J_{\tilde{s}}(0) + \varepsilon J'_{\tilde{s}}(0) + \frac{\varepsilon^2}{(2\pi)^2}\int\int_{-W}^{W}\int_{-W}^{W}\delta\tilde{s}(\omega)$

$\cdot\delta\tilde{s}(\tilde{\omega})\tilde{R}^{-1}(\tilde{\omega})e^{j(\omega+\tilde{\omega})t}d\omega\, d\tilde{\omega}\, dt - \frac{\varepsilon^2\lambda}{(2\pi)^2}\int\int\int_{-W}^{W}\int_{-W}^{W}\delta\tilde{s}(\omega)\ \delta\tilde{s}(\tilde{\omega})\tilde{R}^{-1}(\omega)\tilde{R}^{-1}(\tilde{\omega})$

$e^{j(\omega t+\tilde{\omega}u)} R(t-u)\, d\omega\, d\tilde{\omega}\, dt\, du \leq J_{\tilde{s}}(0)$ for large positive λ, in view of the hypothesis. Thus the condition $s(t) = [\sum_{i=1}^{n} c_i \operatorname{sinc}(W(\cdot-t_i))*R(\cdot)](t)$ for some constants $\{c_i\}_{i=1}^{n}$ is both necessary and sufficient. The $\{c_i\}_{i=1}^{n}$ are chosen, of course, to satisfy the constraint $s(t_i) = s_i$ for $i = 1, 2, \ldots, n$, which is possible since $\det[(\operatorname{sinc}(W(\cdot-t_i)*R(\cdot))(t_j)]_{i,j} \neq 0$. Finally we note that $\inf_{s\in S} \sup_{h\in H} R(s, h)$ is realized by choosing h corresponding to the above s, i.e. $\tilde{H}(\omega)$ is almost everywhere given by

$$\tilde{H}(\omega) = \begin{cases} \frac{\pi}{W} \sum_{i=1}^{n} c_i\, I_{[-W,\, W]}(\omega) e^{-j\omega t_i} & \text{if } \omega \in [-W, W] \\ 0 & \text{otherwise} \end{cases}.$$

Thus $H(t) = \sum_{i=1}^{n} c_i \operatorname{sinc}(W(t-t_i))$, which completes the proof. QED

It might be of interest to determine just how robust the filter of Theorem 1 is. We do know the filter is robust in the minimax sense, but we might in practice desire further knowledge of the sensitivity of the performance of the filter to perturbations in the imperfectly known signal. This is addressed by the following result.

<u>Theorem 2:</u> Let $h(\cdot)$ be the filter of Theorem 1. Then $R(s, h)$ is independent of $s \in S$.

<u>Proof:</u> Note that it suffices to show $\int s(t)h(T-t)dt$ is independent of $s \in S$. Now for each i, $\int s(t) \operatorname{sinc}(W(t-t_i))dt = \frac{1}{4W\pi}\iint_{-W}^{W} \tilde{s}(\omega)e^{j\omega t}d\omega \int_{-W}^{W} e^{(-t_i\tilde{\omega}+\tilde{\omega}t)j}\, d\tilde{\omega}\, dt = \frac{1}{2W}\int_{-W}^{W} \tilde{s}(\omega)\, e^{jt_i\omega}\, d\omega = \frac{\pi}{W}\, s(t_i) = \frac{\pi}{W}\, s_i$. Thus, $\int s(t)h(T-t)dt = \frac{\pi}{W}\sum_{i=1}^{n} c_i\, s_i$, which establishes the desired result. QED

We therefore can see that in view of Theorem 2 the filter of Theorem 1 is indeed very robust to inexact knowledge of the signal $s(\cdot)$, as long as $s(\cdot) \in S$. This filter therefore accomplishes our goal of desensitizing the performance of the filter to inexact knowledge of the signal. The other goal was to accomplish the desensitization in a manner which resulted in

good performance; in particular, we would hope that the performance would be at least as good (and often better) than that of the discrete time filter. This is illustrated by the following result:

Theorem 3: Suppose $R(\cdot)$ is differentiable in a neighborhood of t_i for each i. Suppose the output signal to noise power ratio of the discrete time matched filter exists and is denoted by $\mathcal{R}_0$, and let $h(\cdot)$ denote the (continuous time) filter of Theorem 1. Then $\mathcal{R}(s, h) \geq \mathcal{R}_0$ for all $s \in \mathcal{S}$.

Proof: For each positive real number Δt, let $P_{i,\Delta t}(t) = \frac{1}{\Delta t} \cdot I_{[t_i, t_i+\Delta t]}(t)$. Throughout the remainder of this proof assume Δt is small enough so that for each i, $[t_i, t_i+\Delta t]$ is contained in the neighborhood of t_i for which $R(\cdot)$ is differentiable. Note that in view of Theorem 2, it suffices to show $\mathcal{R}(s, h) \geq \mathcal{R}_0$ for a particular $s \in \mathcal{S}$. Moreover, since $h(\cdot)$ maximizes performance over $\mathcal{H}$ for a particular $s \in \mathcal{S}$, it suffices to show that there exists a sequence of filters $h_K \in \mathcal{H}$ such that $\mathcal{R}(s, h_K) \to \mathcal{R}_0$, where henceforth $s(\cdot)$ corresponds to $h(\cdot)$ according to Theorem 1. Let $h_{\Delta t}(t) = \sum_{i=1}^{n} s_i P_{i,\Delta t}(T-t)$.

Since $h_{\Delta t}(\cdot) \in \mathcal{H}$ for all Δt sufficiently small, it therefore suffices to show $\lim_{\Delta t \to 0} \mathcal{R}(s, h_{\Delta t}) = \mathcal{R}_0$. Now $\int s(t) h_{\Delta t}(T-t)dt = \sum_{i=1}^{n} s_i \int_{t_i}^{t_i+\Delta t} s(t) \cdot \frac{1}{\Delta t}\, dt$,

and it thus follows that $\lim_{\Delta t \to 0} \int s(t) h_{\Delta t}(T-t)dt = \sum_{i=1}^{n} s_i^2$. Moreover, $\iint R(t-u) h_{\Delta t}(T-t) h_{\Delta t}(T-u)dt\, du = \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \int_{t_i}^{t_i+\Delta t} \int_{t_j}^{t_j+\Delta t} R(t-u) \cdot \frac{1}{(\Delta t)^2}\, dt\, du$, and the conditions of the hypothesis allow the use of Leibnitz's rule, which implies

$\lim_{\Delta t \to 0} \iint R(t-u) h_{\Delta t}(T-t) h_{\Delta t}(T-u)dt\, du = \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j R(t_i - t_j)$. We therefore conclude $\lim_{\Delta t \to 0} \mathcal{R}(s, h_{\Delta t}) \to \mathcal{R}_0$, which establishes the desired result.

QED

Note that the additional condition for Theorem 3 of $R(\cdot)$ differentiable in a neighborhood of t_i for all i is quite mild. The continuous time filter in fact often outperforms the discrete time filter. For example, if $R(t) = \text{sinc}(\tilde{W}t)$, where $\tilde{W} > W$, n=2, and $t_2 - t_1 = \pi m/\tilde{W}$ for some positive integer m, then it is straightforward to show that the continuous time

filter outperforms the discrete time filter for all $(s_1, s_2) \neq (0, 0)$.

III. CONCLUSION

We have introduced a method to improve the output signal to noise power ratio of a discrete time matched filter by employing a continuous time filter. This method exploits the bandlimited nature of the unknown continuous time signal and is robust to this inexact knowledge. We have furthermore showed that the method results in an output signal to noise power ratio which upper bounds that of the discrete time filter.

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